Unit - V

Pointwise Convergence of Sequences of Functions

Let $\{f_n\}$ be the sequences of real or complex valued functions having a common domain on the real line R or the complex plane C. For each x in the domain we can form another sequence $\{f_n(x)\}$ whose terms are the corresponding function values.

Let S denote the set x for which the second sequence $\{f_n(x)\}$ converges.

The function f defined by the equation

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ if } x \in S$$

is called the limit function of the sequence $\{f_n\}$ and we say that $\{f_n\}$ converges pointwise to f on the set S.

Examples of Sequence of Real Valued Functions:

Properties of Sequence of Functions:

1. continuity 2. Integrability 3. Differentiability pointwise convergence does not preserve all the above property.

Continuity

Example : 1

A Sequence of continuous function with (a discontinuous limit function

we shall S.T If f_n is continuous at c then f is not necessarily continuous at c

$$\text{Let } f_n(x) = \frac{x^{2n}}{1+x^{2n}} \text{ if } x \in R,\ n=1,2 \ldots$$

The graphs of a few terms are shown in the following figure.

For $-1 < x < 1$ we have

$$S_n - S = \sum_{k=0}^{n} a_k - S.$$

$= f(x) + \sum_{k=0}^{n} a_k - s - f(x)$

$= f(x) - s + \sum_{k=0}^{n} a_k - \sum_{k=0}^{\infty} a_k x^k$

$= f(x) - s + \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} a_k x^k - \sum_{k=n+1}^{\infty} a_k x^k$

Now let $x \in (0,1)$

Then $\frac{1-x^k}{1-x} = 1 + x + x^2 + \cdots + x^{k-1}$

$(1-x^k) = (1 + x + x^2 + \cdots + x^{k-1})(1-x)$

$\leq k(1-x)$

$|1-x^k| < |1-x||k| = (1-x)|k|$

For $n \geq N$ and $0 < x < 1$,

$|s_n - s| \leq |f(x) - s| + \sum_{k=0}^{n} |a_k||1-x^k| + \sum_{k=n+1}^{\infty} |a_k||1-x^k|$

$< \epsilon/3 + (1-x) \sum_{k=0}^{n} k|a_k| + \sum_{k=n+1}^{\infty} |a_k| x^k$

$|s_n - s| < \epsilon/3 + (1-x) n \sigma_n + \frac{\epsilon}{3n} \cdot \frac{x^{n+1}}{1-x}$

$\sum_{k=n+1}^{\infty} |a_k| x^k < \sum_{k=n+1}^{\infty} \frac{\epsilon}{3n} x^k$

$= \frac{\epsilon}{3n} (x^{n+1} + x^{n+2} + \cdots)$

$= \frac{\epsilon}{3n} x^{n+1} (1 + x + \cdots)$ Also if $x < 1$, $x^{n+1} < 1$

$\Rightarrow |s_n - s| < \epsilon/3 + (1-x) n \sigma_n + \epsilon/3n \left(\frac{1}{1-x}\right)$

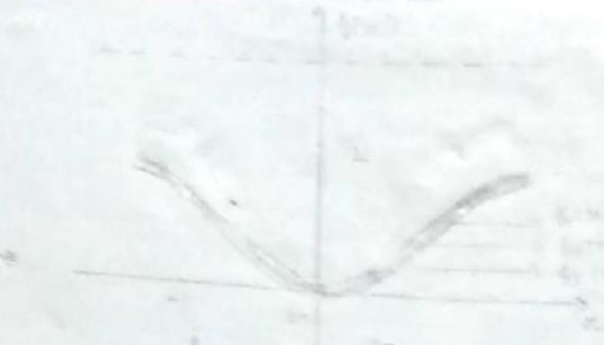
Taking $x = x_n = 1 - 1/n$

$|s_n - s| < \epsilon/3 + 1/\cancel{n}\,\cancel{n}\sigma_n + \epsilon/3\cancel{n} \cdot \cancel{n}$

$< \epsilon/3 + \epsilon/3 + \epsilon/3$

$< \cancel{3}\epsilon/\cancel{3}$

$\Rightarrow |s_n - s| < \epsilon$



$f_n(x) = \frac{x^{2n}}{1+x^{2n}}$, $n = 1, 2, 3, \ldots$

Case (i)

Let $|x| < 1$ then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}} = 0$$

$\because |x| < 1$ if $x = 1/2$ then $(1/2)^{2n} \to 0$ as $n \to \infty$

Case (ii)

Let $|x| = 1$ then $f_n(x) = 1/2$

$$\lim_{n\to\infty} f_n(x) = 1/2$$

Case (iii)

Let $|x| > 1$ then $\frac{1}{x^{2n}} \to 0$

$$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}} \left(= \lim_{n\to\infty} \frac{x^{2n}}{x^{2n}\left(1+\frac{1}{x^{2n}}\right)} \right)$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{x^{2n}}} = \frac{1}{1+0}$$

$$\lim_{n\to\infty} f_n(x) = 1 \text{ when } |x| > 1$$

The Sequence $\{f_n\}$ converges pointwise to the point function f on R where

$$f(x) = \begin{cases} 0 & \text{if } |x| < 1, \text{ i.e. } -1 < x < 1 \\ 1/2 & \text{if } |x| = 1, \text{ i.e. } x = \pm 1 \\ 1 & \text{if } |x| > 1, \text{ i.e. } x > 1 \text{ (or) } x < -1 \end{cases}$$

$f(1-) = 0$, $f(1) = 1/2$, $f(1+) = 1$

f is discontinuous at $x = 1$

IIIly f is discontinous at $x = -1$

Thus each f_n is continuous on R but f is discontinuous at $x = 1$ and $x = -1$.

Note: Suppose g is continuous then

$$\lim_{x \to c} \left\{ \lim_{n \to \infty} f_n(x) \right\} = \lim_{x \to c} f(x)$$

$$= f(c)$$

$$\lim_{n \to \infty} \left\{ \lim_{x \to c} f_n(x) \right\} = \lim_{n \to \infty} f_n(c)$$

$$= f(c)$$

$$\lim_{x \to c} \left\{ \lim_{n \to \infty} f_n(x) \right\} = \lim_{n \to \infty} \left\{ \lim_{x \to c} f_n(x) \right\}$$

Integrability

Example : 2

A Sequence of functions for which

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \lim_{n \to \infty} f_n(x)\, dx$$

T.S.T

Let $f_n(x) = n^2 x (1-x)^n$ if $x \in R$, $n = 1, 2 \ldots$

If $0 \le x \le 1$

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} n^2 x (1-x)^n$$

$$= 0$$

$$\Rightarrow \int_0^1 \lim_{n \to \infty} f_n(x)\, dx = \int_0^1 0\, dx$$

$$\int_0^1 \lim_{n \to \infty} f_n(x)\, dx = 0 \longrightarrow (1)$$

Consider, $\int_0^1 f_n(x)\, dx = \int_0^1 n^2 x (1-x)^n\, dx$

$$= \int_1^0 n^2 (1-t) t^n (-dt)$$

Put $1-x = t$

$dx = dt$

$$= n^2 \int_0^1 (t^n - t^{n+1})\, dt$$

when $x=0$, $t=1$

when $x=1$, $t=0$

$$= n^2 \left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right]_0^1$$

$$= n^2 \left[\frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$\int_0^1 f_n(x)\,dx = n^2\left[\frac{n+2-n-1}{(n+1)(n+2)}\right]$$

$$= n^2\left[\frac{1}{(n+1)(n+2)}\right]$$

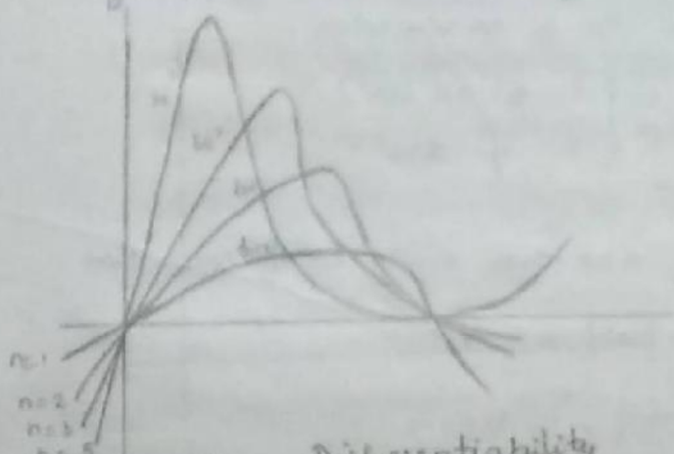
$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n^2}{n^2(1+1/n)(1+2/n)}$$

$$= 1$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 1 \longrightarrow (2)$$

From (1) and (2)

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$$



Example : 3 Differentiability

A sequence of differentiable functions $\{f_n\}$ with limit for which $\{f_n'\}$ diverges

Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ if $x \in \mathbb{R}$, $n = 1, 2 \ldots$

$$\lim_{n\to\infty} f_n(x) = 0$$

$$\Rightarrow \frac{d}{dx}\lim_{n\to\infty} f_n(x) = 0 \longrightarrow (1)$$

Consider, $\dfrac{d}{dx} f_n(x) = \dfrac{d}{dx}\left(\dfrac{\sin nx}{\sqrt{n}}\right)$

$$= \frac{n\cos nx}{\sqrt{n}}$$

$$= \sqrt{n}\cos nx$$

$$\lim_{n\to\infty}\frac{d}{dx} f_n(x) = \lim_{n\to\infty}\sqrt{n}\cos nx = \infty$$

$\therefore \lim_{n\to\infty} f_n'(x)$ does not exist for any x

$$\therefore \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty}\left(\frac{d}{dx} f_n(x)\right)$$

Definition of uniform convergence

Pointwise Convergence:

Let $\{f_n\}$ be a sequence of functions which converges pointwise on a set S to a limit function f and for each $\epsilon>0$, there exists an N (depending on both x and ϵ) such that

$n>N$ implies $|f_n(x) - f(x)| < \epsilon$ for every $x \in S$

Example:

Let $f_n(x) = x^n$, $x \in [0,1]$ $n=1,2\ldots$

$\{f_n\}$ converges to f where where

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x=1 \end{cases}$$

definition,

for every $\epsilon>0$ there exists an N such that

$|f_n(x) - f(x)| < \epsilon \ \forall \ n>N$

$\epsilon = 1/2$ $|x^n - f(x)| < 1/2 \ \forall \ n>N$

i.e) The above inequality is true when $N=1$ and for $x=0,1$

if $x=3/4$ and $\epsilon = 1/2$ the inequality is not true when $N=1$ and $N=2$

But it is true when $N=3$

i.e) $|(3/4)^n - 0| < 1/2$ when $N=2$

$\therefore$ The choice of N depends on both x and ϵ

Uniform Convergence

A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S if, for every $\epsilon>0$, there exists an N (depending only on ϵ) such that $n>N$ implies

$|f_n(x) - f(x)| < \epsilon$, for every $x \in S$ denote this symbolically by $f_n \to f$ uniformly on S.

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Note:

when each term of the Sequence $\{f_n\}$ is real valued the geometrical interpretation of uniform convergence is as follows

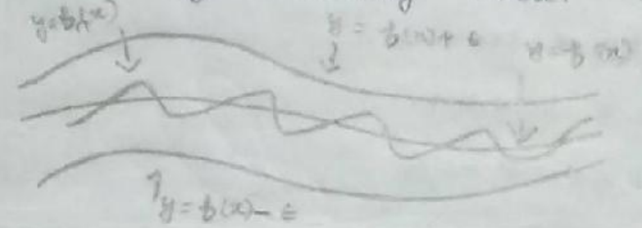
$$|f_n(x) - f(x)| < \epsilon$$

$$\Rightarrow -\epsilon < f_n(x) - f(x) < \epsilon$$

$$\Rightarrow f(x) - \epsilon < f_n(x) < f(x) + \epsilon \longrightarrow (1)$$

If (1) is to hold for all $n > N$ and for all $x \in S$,

" The entire graph of f_n that is the set, $\{(x, y) : y = f_n(x) ; x \in S\}$ lies within a 'band' of height 2ϵ situated symmetrically about the graph of f "



Note: uniform Convergence implies pointwise Convergence, but the converse is not true.

Uniformly Bounded

A Sequence $\{f_n\}$ is said to be uniformly Bounded on S, if there exists a constants $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \text{ in } S \text{ and}$$

Eg:- If $f_n(x) = \sin nx$, $x \in [0, \pi]$

$$|f_n(x)| < 1$$

$\therefore \{f_n\}$ is uniformly bounded by 1.

Result:

If each function f_n is bounded and if $f_n \to f$ uniformly on S, then $\{f_n\}$ is uniformly bounded on S

proof:- given, (i) Each f_n is bounded

(ii) $f_n \longrightarrow f$ uniformly on S

T.P.T $\{f_n\}$ is uniformly bounded on S

Since $f_n \to f$ uniformly on S.

For every $\epsilon > 0$, there exists $N > 0$ such that

$n \geq N$ implies $|f_n(x) - f(x)| < \epsilon \ \forall \ x \in S$

Now, $|f_n(x)| - |f(x)| \leq |f_n(x) - f(x)| < \epsilon$

$\Rightarrow |f_n(x)| - |f(x)| < \epsilon$

$\Rightarrow |f_n(x)| < |f(x)| + \epsilon \longrightarrow (1)$

Let $\epsilon = 1$, then

$|f_n(x)| < |f(x)| + 1 \ \forall \ n \geq N$ and $\forall \ x \in S$

Since each f_n is bounded.

f is bounded

$\Rightarrow |f(x)| \leq M \ \forall \ x \in S$

Sub in (1) we have

$|f_n(x)| < M + 1 \ \forall \ n > N$ and $\forall \ x \in S$

Since each function f_n is bounded.

$f_1, f_2, \ldots f_{N-2}, f_{N-1}$ are all bounded functions on S

Then there exists constants $M_1, M_2, \ldots M_{N-1}, M_N \ni$

$|f_i(x)| \leq M_i \quad , i = 1, 2, \ldots N-2, N-1$

Let ie) $M_n k \{M_1 M_2 \ldots M_{N-1}, M_N\}$

then $|f_n(x)| \leq k \ \forall \ x \in S$ and $\forall \ n$

Thus $\{f_n\}$ is uniformly bounded on S.

Uniform Convergence and continuity:

Theorem: 9.2 Uniform limit theorem:

Assume that $f_n \to f$ uniformly on S If each f_n is continuous at a point c of S, then the limit function f is also continuous at c.

Proof :

Given

(i) $f_n \to f$ uniformly on S

(ii) Each f_n is continuous at a point c of S.

T.P.T The limit function f is also continuous at c

If c is an isolated point of S then f is automatically continuous at c

So assume that c is an accumulation point of S.

Then we have to p.T f is continuous at c

(i.e) T.P.T $\lim_{x \to c} f(x) = f(c)$

Since $f_n \to f$ uniformly on S,

for every $\epsilon > 0$ there exists an M (depending only on ϵ) such that,

$n \geq M$ implies $|f_n(x) - f(x)| < \epsilon/3 \quad \forall x \in S \longrightarrow (1)$

Since each f_n is continuous at c, f_M is continuous at c

$\therefore$ for any given $\epsilon > 0$, there exists a neighbourhood $B(c)$ such that,

$x \in B(c) \cap S \Rightarrow |f_M(x) - f_M(c)| < \epsilon/3$

~~For $x \in B(c) \cap S$~~

Now, $|f(x) - f(c)| = |f(x) - f_M(x) + f_M(x) - f_M(c) + f_M(c) - f(c)|$

$\leq |f_M(x) - f(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$

$\Rightarrow |f(x) - f(c)| < \epsilon$

$\therefore$ limit function f is continuous at c.

Note : 1

If c is an accumulation point of S, the conclusion of the above theorem implies that,

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x)$$

Note: 2

Uniform convergence of $\{f_n\}$ is sufficient but not necessary to transmit continuity from the individual terms to the limit function.

Theorem: The Cauchy condition for uniform convergence

Let $\{f_n\}$ be a sequence of functions defined on a set S. There exists a function f such that $f_n \to f$ uniformly on S if and only if the following condition is satisfied.

For every $\epsilon > 0$ there exists an N such that $m > N$ and $n > N$ implies

$$|f_m(x) - f_n(x)| < \epsilon \text{ for every } x \text{ in } S.$$

Proof:

Given $\{f_n\}$ is a sequence of functions defined on a set S

Necessary part:

Assume that there exists a function f such that $f_n \to f$ uniformly on S

T.P.T Cauchy condition is satisfied

Since $f_n \to f$ uniformly on

Given $\epsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall\ n \geq N \quad \text{or} \qquad \to (1)$$

Taking $m > N$, we have

$$|f_m(x) - f(x)| < \epsilon/2 \quad \forall\ x \in S$$

For $m > N$, $n > N$

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)|$$

$$< |f_m(x) - f(x)| + |f_n(x) - f(x)|$$

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